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AN OPTIMAL THEOREM FOR THE SPHERICAL MAXIMAL OPERATOR ON THE HEISENBERG GROUP

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ABSTRACT

Let $\mathbb{I}^n = \mathbb{C}^n \times \mathbb{R}$ be the Heisenberg group and μ_r be the normalized surface measure on the sphere of radius r in $\mathbb{C}^n$. Let $Mf = \sup_{r>0} |f * \mu_r|$. We prove an optimal L^p-boundedness result for the spherical maximal function Mf, namely we prove that M is bounded on $L^p(\mathbb{I}^n)$ if and only if $p > \frac{2n}{2n-1}$.

1. The main results

Let $H^n = \mathbb{C}^n \times \mathbb{R}$ be the $(2n+1)$-dimensional Heisenberg group with the group law

$$(z,t)(w,s) = \Big(z + w, t + s + \frac{1}{2}\operatorname{Im} z \cdot \bar{w}\Big).$$

Given a function f on H^n consider the spherical means

$$f * \sigma_r(z,t) = \int_{|w|=r} f\Big(z - w, t - \frac{1}{2}\operatorname{Im} z \cdot \bar{w}\Big) d\sigma_r(w) \tag{1.1}$$

where σ_r is the normalised surface measure on the sphere $S_r = \{(z,0) : |z| = r\}$ in H^n. In [2] Nevo and the second author studied the maximal and pointwise

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ergodic theorems in L^p for these spherical means. They showed that $\{\sigma_r\}$ is a pointwise ergodic family in L^p for every ergodic action of H^n, $n \geq 2$ on a probability space (see [2] for the relevant definitions) for all $p > (2n-1)/(2n-2)$. For actions of the reduced Heisenberg group the range of p was extended to $p > 2n/(2n-1)$ and it was conjectured that the same should be true for the full group H^n.

A basic ingredient in the proof of the ergodic theorem is the L^p boundedness of the maximal function

$$M_\sigma f(z,t) = \sup_{r>0} |f * \sigma_r(z,t)| \tag{1.2}$$

associated to the spherical means. In [2] it was shown that M_σ is bounded on $L^p(H^n)$ for all $p > (2n-1)/(2n-2)$. In this paper we establish the optimal result, namely

THEOREM 1.1: *Let $n \geq 2$. Then the maximal operator defined by (1.2) is bounded on $L^p(H^n)$ if and only if $p > 2n/(2n-1)$.*

This is the analogue of the celebrated spherical maximal theorem of Stein [3] on $\mathbb{R}^n$. Note that the spherical means σ_r are averages over the sphere S_r which is of co-dimension 2 and hence more singular. As a consequence of Theorem 1.1 we obtain the following result.

THEOREM 1.2: *Let $n \geq 2$. Then $\{\sigma_r\}$ is a pointwise ergodic family in L^p for all $p > 2n/(2n-1)$.*

The following remarks are in order. The main result, namely Theorem 1.1, has been recently proved by Mueller and Seeger [1] using different methods. Actually, they have extended the above result to a more general setting of surfaces and to a class of step two nilpotent groups including all H-type groups. They use Fourier integral operators to study the maximal function whereas we prove Theorem 1.1 by modifying the arguments presented in [2] and combining it with Stein's original square function method. Our proof of Theorem 1.1 is also valid in the general set-up of H-type groups. Though the results in [1] are more general, our approach may be of independent interest as it does not appeal to the theory of Fourier integral operators.

The spherical means $f * \sigma_r$ are naturally associated to the Gelfand pair $(H^n, U(n))$ where $U(n)$ is the unitary group. For K-spherical means associated to other Gelfand pairs (H^n, K) and the associated maximal functions we refer to [7].

We closely follow the notations set up in [2] and [6] and we refer to these for any unexplained terminology.

2. Proof of Theorem 1.1

We only need to prove Theorem 1.1 as Theorem 1.2 can be deduced as in [2]. The starting point is the following expression for $f * \sigma_r$ which has been already used in [2]:

$$f * \sigma_r(z,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Big(\sum_{k=0}^{\infty}\psi_k^{n-1}(\sqrt{|\lambda|}r)f^\lambda *_\lambda \varphi_k^\lambda(z)\Big)e^{i\lambda t}|\lambda|^n d\lambda. \tag{2.1}$$

The notations used in this equation are as follows. For a function f on H^n, $f^\lambda(z)$ stands for the partial inverse Fourier transform

$$f^\lambda(z) = \int_{-\infty}^{\infty} e^{i\lambda t} f(z,t)dt; \tag{2.2}$$

the λ-twisted convolution $f^\lambda *_\lambda \varphi_k^\lambda(z)$ is the one given by

$$f^\lambda *_\lambda \varphi_k^\lambda(z) = \int_{\mathbb{C}^n} f^\lambda(z-w)\varphi_k^\lambda(w)e^{i\frac{\lambda}{2}\operatorname{Im} z\cdot\bar{w}}dw; \tag{2.3}$$

$\varphi_k^\lambda(z)$ is the (dilated) Laguerre function

$$\varphi_k^\lambda(z) = L_k^{n-1}\Big(\frac{1}{2}|\lambda||z|^2\Big)e^{-\frac{1}{4}|\lambda||z|^2}$$

and $\psi_k^{n-1}(r)$ or, more generally, $\psi_k^\alpha(r)$ is the normalised Laguerre function

$$\psi_k^\alpha(r) = \frac{\Gamma(k+1)\Gamma(\alpha+1)}{\Gamma(k+\alpha+1)}L_k^\alpha\Big(\frac{1}{2}r^2\Big)e^{-\frac{1}{4}r^2}. \tag{2.4}$$

The Laguerre functions ψ_k^α are defined even for complex α with $\operatorname{Re}\alpha > -1$. We consider the following analytic family of operators:

$$M_r^\alpha f(z,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Big(\sum_{k=0}^{\infty}\psi_k^{\alpha}(\sqrt{|\lambda|}r)f^\lambda *_\lambda \varphi_k^\lambda(z)\Big)e^{-i\lambda t}|\lambda|^n d\lambda. \tag{2.5}$$

A priori it is not clear if this operator is well defined or not, but we will say more about that in a moment. We require the following formula connecting Laguerre polynomials of different type:

$$L_k^\alpha(r) = \frac{\Gamma(k+\alpha+1)}{\Gamma(\alpha-\beta)\Gamma(k+\beta+1)}\int_0^1 t^\beta(1-t)^{\alpha-\beta-1}L_k^\beta(rt)dt, \tag{2.6}$$

valid for $\operatorname{Re}(\alpha-\beta)>0$ and $\operatorname{Re}\beta>-1$. Defining

$$P_r f(z,t)=\int_{-\infty}^{\infty} e^{-i\lambda t}e^{-\frac{1}{4}|\lambda| r} f^{\lambda}(z)d\lambda \tag{2.7}$$

to be the Poisson integral of f in the t-variable we have

$$M_r^{\alpha} f(z,t)=\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\beta)\Gamma(\beta+1)}\int_0^1 t^{2\beta+1}(1-t^2)^{\alpha-\beta-1}(P_{r^2(1-t^2)}M_{rt}^{\beta} f)(z,t)dt. \tag{2.8}$$

We want to use Stein's analytic interpolation theorem to the maximal functions

$$M_*^{\alpha} f(z,t)=\sup_{r>0}|M_r^{\alpha} f(z,t)|$$

associated to this family of operators.

More precisely we would like to prove that $M_*^{\alpha} f$ is bounded on $L^p(H^n)$, $1<p<\infty$ when $\operatorname{Re}\alpha=n+\delta$ and on $L^2(H^n)$ when $\operatorname{Re}\alpha=\delta$, for every $\delta>0$. As $M_r^{n-1}f=f*\sigma_r$ by analytic interpolation we can obtain Theorem 1.1. For more details of this argument we refer to Stein–Wainger [3].

Before addressing the question of boundedness of $M_*^{\alpha} f$ let us see why these operators are well defined. Actually, for $\operatorname{Re}\alpha=n+i\gamma+\delta$ there is no problem in defining $M_r^{\alpha} f$ as

$$M_r^{\alpha} f(z,t)=\frac{\Gamma(n+i\gamma+\delta)}{\Gamma(1+\delta+i\gamma)\Gamma(n)}\int_0^1 s^{2n-1}(1-s^2)^{i\gamma+\delta}P_{r^2(1-s^2)}M_{rs}^{n-1} f(z,t)ds. \tag{2.9}$$

Indeed, $M_{rt}^{n-1} f=f*\sigma_{rt}$ is in $L^p(H^n)$ when $f\in L^p(H^n)$ and the same is true of $P_r f$. Hence $M_r^{\alpha} f$ makes sense as an L^p function. However, there is trouble at the other end, viz. when $\operatorname{Re}\alpha=-\frac{1}{2}+\delta$.

In view of the spectral theorem for $M_r^{\alpha} f$, with $\operatorname{Re}\alpha=-\frac{1}{2}+\delta$ to make sense as an L^2 function we require that $\{\psi_k^{\alpha}(r)\}$ is a uniformly bounded family of functions. But it is known (see Szego [4]) that such a uniform estimate holds only for $\operatorname{Re}\alpha>-\frac{1}{3}$. Fortunately, it is also known that

$$\sup_{k\geq 0}\sup_{0\leq r\leq a}|\psi_k^{\alpha}(r)|\leq c_a \tag{2.10}$$

for all $\operatorname{Re}\alpha\geq-\frac{1}{2}$. Suppose now f is a function which is band-limited in the t-variable. By this we mean that $f^{\lambda}(z)$ is supported in $|\lambda|\leq a$ for some $a>0$. For such a function, $M_r^{\alpha} f$ is well defined as an L^2 function as long as $\operatorname{Re}\alpha>-\frac{1}{2}$. As the class of band-limited functions is dense in $L^p(H^n)$ it is enough to show that

$$\|M_*^\alpha f\|_p \le C_1\|f\|_p, \quad 1<p<\infty, \quad \operatorname{Re}\alpha > n, \tag{2.11}$$

$$\|M_*^\alpha f\|_2 \le C_2\|f\|_2, \quad \operatorname{Re}\alpha > 0, \tag{2.12}$$

where the constants C_1 and C_2 are independent of the supports of f^λ. It is while proving (2.12) that we need to use the operators M_r^α for $\operatorname{Re}\alpha > -\frac{1}{2}$.

The estimate (2.11) can be easily dealt with. In view of (2.9), since P_r and M_r^{n-1} commute, we have

$$M_r^{n+i\gamma+\delta} f(z,t) = \frac{\Gamma(n+i\gamma+\delta)}{\Gamma(1+\delta+i\gamma)\Gamma(n)} \int_0^1 s^{2n-1}(1-s^2)^{i\gamma+\delta}(P_{r^2(1-s^2)}f) * \sigma_{rs}(z,t)ds. \tag{2.13}$$

Making a change of variables and noting that $|P_s f|$ is dominated by the Hardy–Littlewood maximal function Λf in the t-variables, we get

$$|M_r^{n+i\gamma+\delta} f(z,t)| \le C(\gamma) r^{-2n} \int_0^r s^{2n-1}(\Lambda f) * \sigma_s(z,t)ds$$

where $C(\gamma)$ is of admissible growth. This shows that

$$|M_r^{n+i\gamma+\delta} f(z,t)| \le C(\gamma)\frac{1}{r}\int_0^r \Lambda f * \sigma_s(z,t)ds. \tag{2.14}$$

But now the right-hand side can be realised as Birkhoff averages over the group of reals as was done in [2], see Proposition 4.2. We can therefore appeal to the strong maximal inequality for $\mathbb{R}$-actions to conclude that $M_*^{n+i\gamma+\delta} f$ is bounded on L^p, $1<p<\infty$.

In order to prove (2.12) we bring in the square function. First observe that (2.8) leads to the estimate

$$|M_r^\alpha f(z,t)|^2 \le C(\alpha,\beta)\int_0^1 |P_{r^2(1-s^2)} M_{rs}^\beta f(z,t)|^2 ds \tag{2.15}$$

where

$$C(\alpha,\beta) = \Big|\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\beta)\Gamma(\beta+1)}\Big| \int_0^1 s^{4\beta+2}(1-s^2)^{2\operatorname{Re}\alpha-2\beta-2}ds,$$

which is finite and has admissible growth provided $\beta > -\frac{3}{4}$ and $\operatorname{Re}\alpha > \beta+\frac{1}{2}$. The right-hand side of (2.15) is dominated by the sum of

$$\Big(\int_0^1 |P_{r^2-r^2s^2}(M_{rs}^\beta f - M_{rs}^n f)|^2 ds\Big)^{\frac{1}{2}} \tag{2.16}$$

and

$$\left(\int_0^1 |P_{r^2-s^2r^2} M_{rs}^n f|^2 ds \right)^{\frac{1}{2}}. \tag{2.17}$$

We first look at the second term which is

$$\left(\frac{1}{r} \int_0^r |P_{r^2-s^2} M_s^n f|^2 ds \right)^{\frac{1}{2}}.$$

In view of (2.13), we have

$$P_{r^2-s^2} M_s^n f = M_s^n P_{r^2-s^2} f$$

is given by the integral

$$\int_0^1 u^{2n-1} P_{s^2-s^2u^2}(P_{r^2-s^2} f) * \sigma_{su} du = \int_0^1 u^{2n-1} P_{r^2-s^2u^2} f * \sigma_{su} du.$$

This equation leads to the estimate

$$|P_{r^2-s^2} M_s^n f| \le C \int_0^1 u^{2n-1} \Lambda f * \sigma_{su} du$$

which in turn gives

$$\begin{aligned} \frac{1}{r} \int_0^r |P_{r^2-s^2} M_s^n f|^2 ds &\le C \frac{1}{r} \int_0^r \left(\int_0^s u^{2n-1} \Lambda f * \sigma_u du \right)^2 s^{-4n} ds \\ &\le C \frac{1}{r} \int_0^r \left(\frac{1}{s} \int_0^s \Lambda f * \sigma_u du \right)^2 ds \\ &\le C \sup_{s>0} \left(\frac{1}{s} \int_0^s \Lambda f * \sigma_u du \right)^2. \end{aligned}$$

As before, the last term defines a bounded operator on $L^p(H^n)$, $1 < p < \infty$ and hence the maximal function associated to (2.17) has the same property.

It remains to prove the L^2 boundedness of the maximal function

$$\sup_{r>0} \int_0^1 |P_{r^2-r^2s^2}(M_{rs}^\beta f - M_{rs}^n f)|^2 ds \tag{2.18}$$

for $\beta > -\frac{1}{2}$ which in turn will imply the L^2 boundedness of the maximal function associated to (2.15) for $\operatorname{Re}\alpha > 0$. The above integral is bounded by

$$\begin{aligned} \frac{1}{r} \int_0^r |P_{r^2-s^2}(M_s^\beta f - M_s^n f)|^2 ds &\le \frac{1}{r} \int_0^r (\Lambda(M_s^\beta f - M_s^n f))^2 ds \\ &\le \int_0^\infty (\Lambda(M_s^\beta f - M_s^n f))^2 \frac{ds}{s}. \end{aligned}$$

As Λ is bounded on $L^2(H^n)$ this leads to

$$\begin{aligned}&\int_{H^n}\sup_{r>0}\frac{1}{r}\int_0^r |P_{r^2-r^2s^2}(M_{rs}^\beta f - M_{rs}^n f)(z,t)|^2 ds dz dt\\ &\leq \int_{H^n}\int_0^\infty (\Lambda(M_s^\beta f - M_s^n f)(z,t))^2 \frac{ds}{s} dz dt\\ &\leq C\int_{H^n}\int_0^\infty |M_s^\beta f(z,t) - M_s^n f(z,t)|^2 \frac{ds}{s} dz dt.\end{aligned}$$

Therefore, we are led to prove the L^2 boundedness of the square function

$$(g_\beta(f)(z,t))^2 = \int_0^\infty |M_s^\beta f(z,t) - M_s^n f(z,t)|^2 \frac{ds}{s} \tag{2.19}$$

for $\beta > -\frac{1}{2}$. In view of the Plancherel theorem we only need to prove the uniform bounds

$$\int_0^\infty |\psi_k^\beta(\sqrt{|\lambda|}s) - \psi_k^n(\sqrt{|\lambda|}s)|^2 \frac{ds}{s} \leq C$$

for all $k \geq 0$, $\lambda \in \mathbb{R}$, or equivalently

$$\int_0^\infty |\psi_k^\beta(s) - \psi_k^n(s)|^2 \frac{ds}{s} \leq C \tag{2.20}$$

for all $k \geq 0$ when $\beta > -\frac{1}{2}$. Note that the final estimate is independent of λ, does not depend on the L^∞ bounds of ψ_k^β and hence the bounds we get will not depend on the support of f^λ. First consider the integral

$$\int_0^{k^{-1/2}} |\psi_k^\beta(s) - \psi_k^n(s)|^2 \frac{ds}{s} \leq \int_0^{k^{-1}} |F_\beta(s) - F_n(s)|^2 \frac{ds}{s}$$

where

$$F_\beta(s) = \frac{\Gamma(k+1)\Gamma(\beta+1)}{\Gamma(k+\beta+1)} L_k^\beta(s).$$

Since $F_\beta(0) = F_n(0) = 1$ we can write

$$\begin{aligned}&\left(\int_0^{k^{-1}} |F_\beta(s) - F_n(s)|^2 \frac{ds}{s}\right)^{\frac{1}{2}}\\ &\leq \left(\int_0^{k^{-1}} |F_\beta(s) - F_\beta(0)|^2 \frac{ds}{s}\right)^{\frac{1}{2}} + \left(\int_0^{k^{-1}} |F_n(s) - F_n(0)|^2 \frac{ds}{s}\right)^{\frac{1}{2}}.\end{aligned} \tag{2.21}$$

By the mean value theorem, for any $0 < s \leq 1/k$

$$|F_\beta(s) - F_\beta(0)| \leq s \sup_{0<t\leq 1} |F_\beta'(t)|.$$

Now we know that (see Szego [4])

$$F'_\beta(t) = -\frac{\Gamma(k+1)\Gamma(\beta+1)}{\Gamma(k+\beta+1)} L_{k-1}^{\beta+1}(t)$$

and so $\sup_{0\le t\le 1} |F'_\beta(t)| \le ck$ using the estimate on Laguerre polynomials. Hence

$$\int_0^{1/k} |F_\beta(s) - F_\beta(0)|^2 \frac{ds}{s} \le ck^2 \int_0^{1/k} sds \le C.$$

The second term in (2.21) is bounded similarly.

Next we estimate the integral

$$\int_{k^{-\frac{1}{2}}}^\infty |\psi_k^\beta(s)|^2 \frac{ds}{s} = \frac{1}{2} \int_{k^{-1}}^\infty |\psi_k^\beta(\sqrt{s})|^2 \frac{ds}{s}.$$

Let $\mathcal{L}_k^\beta(s)$ be the normalised Laguerre functions defined by

$$\mathcal{L}_k^\beta(s) = \Big(\frac{\Gamma(k+1)\Gamma(\beta+1)}{\Gamma(k+\beta+1)}\Big)^{\frac{1}{2}} L_k^\beta(s) e^{-s/2} s^{\beta/2}.$$

In terms of $\mathcal{L}_k^\beta(s)$ we have

$$\psi_k^\beta(\sqrt{s}) = \Big(\frac{\Gamma(k+1)\Gamma(\beta+1)}{\Gamma(k+\beta+1)}\Big)^{\frac{1}{2}} \mathcal{L}_k^\beta(s) s^{-\beta/2}.$$

The estimates given in [5], Lemma 1.5.3 show that

$$\begin{aligned}
&|\psi_k^\beta(\sqrt{s})| \le C(sk)^{-1/4-\beta/2}, \quad 1/k \le s \le k/2,\\
&|\psi_k^\beta(\sqrt{s})| \le C(sk)^{-\beta/2} k^{-1/4}(k^{1/3} + |k-s|)^{-1/4}, \quad k/2 \le s \le 3k/2,\\
&|\psi_k^\beta(\sqrt{s})| \le Ce^{-\gamma s}(sk)^{-\beta/2}, \quad s \ge 3k/2,
\end{aligned}$$

where γ is independent of k.

Using these estimates we see that

$$\int_{k^{-1}}^{k/2} |\psi_k^\beta(\sqrt{s})|^2 \frac{ds}{s} \le Ck^{-\beta-\frac{1}{2}} \int_{k^{-1}}^{k/2} s^{-\beta-3/2} ds$$

which is bounded if $\beta > -\frac{1}{2}$. Now look at

$$\begin{aligned}
\int_{k/2}^{k-k^{1/3}} |\psi_k^\beta(\sqrt{s})|^2 \frac{ds}{s} &\le Ck^{-2\beta-3/2} \int_{k/2}^{k} (k-s)^{-1/2} ds\\
&\le Ck^{-2\beta-3/2} \int_0^{k/2} s^{-1/2} ds \le Ck^{-2\beta-1}
\end{aligned}$$

which is again bounded for $\beta > -\frac{1}{2}$. The part of the integral taken from $k+k^{1/3}$ to $3k/2$ is also bounded. Finally,

$$\int_{k-k^{1/3}}^{k+k^{1/3}} |\psi_k^\beta(\sqrt{s})|^2 \frac{ds}{s} \leq Ck^{-2\beta-3/2}k^{-1/6}k^{1/3} \leq C.$$

Thus we have proved the uniform estimates (2.20) and hence the g_β-function is bounded on L^2. An appeal to the analytic interpolation theorem completes the proof of Theorem 1.1.

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